On shear flow past flat plates

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The boundary layer on a semi-infinite flat plate placed in a two-dimensional, unbounded, steady, constant shear flow of a viscous incompressible fluid is examined on the basis of the constant-pressure assumption. An asymptotic solution is obtained first for large vorticity numbers. Then an approximate solution is found for arbitrary vorticity numbers that gives good agreement with exact calculations for the extreme cases of small and large vorticity numbers. The present calculations are limited to the boundary layer on the top side of the plate only.

1. Introduction

A problem that has recently stimulated much interest concerns the determination of the flow field about a body that is immersed in the stream of a viscous fluid that contains vorticity generated by some external mechanism other than the body. To study this problem in its essential features, Li (1955) introduced the idealized model of the two-dimensional, unbounded, steady, constant shear flow of an incompressible viscous fluid past an infinitesimally thin, semi-infinite flat plate that is aligned parallel to the oncoming flow. This oncoming flow field is essentially the superposition, at constant pressure P, of a uniform flow with constant velocity U upon a shear flow with a linear velocity distribution ωy, where ω is the positive constant external vorticity and (u, v) are components of velocity in a rectangular (x, y) system. The plate is placed on $y = 0$, $x \geqslant 0$.

Despite the simplicity of the oncoming flow pattern and body geometry (the flow in the wake of an otherwise finite body has been eliminated), this non-linear problem is still extremely difficult to solve without the introduction of reasonable simplifying assumptions. The common approach in the past has been to adopt, with suitable modifications, the approximations of classical boundary-layer theory as the starting-point of an heuristic investigation. As a result, two different models—based on different 'suitable modifications'—have been advanced to predict the flow behaviour. They will be discussed below.

According to the classical theory of Prandtl–Blasius (see Prandtl & Tietjens 1934) for the uniform flow past a semi-infinite flat plate, two basic assumptions were made: (i) the existence of a thin boundary layer on the plate where inertia and viscous forces are of the same order of magnitude, and (ii) this thin boundary layer cannot significantly disturb the mainstream pressure field to the first approximation (or, it represents only a secondary effect on the pressure at most). The theoretical justification for these physical assumptions has been subsequently established *a posteriori* by examining the flow due to displacement thickness.

A somewhat more convincing, but still indirect, justification has been given by the good agreement between the predicted velocity profile across the boundary layer and the measured one for plates of finite length (this good agreement also indicates that the wake flow has a higher-order effect on the plate proper). At any rate, there seems to be no doubt that the basic assumptions of Prandtl–Blasius are good ones for a plate that is semi-infinite in length, at least for the initial approximation. Thus, the pressure distribution along the surface of the plate is given by the inviscid distribution as if the boundary layer was absent, and, by virtue of the constancy of pressure across the thin boundary layer, this inviscid pressure distribution is also effective at the outer edge of the boundary layer (the outer edge of the boundary layer is interpreted here in the asymptotic sense as where the viscous diffusion term normal to the plate in the governing equations becomes exponentially small to within a specified amount). Then, by virtue of Bernoulli's equation, the corresponding velocity distribution at the outer edge of the boundary layer is automatically determined and, moreover, establishes the asymptotic boundary condition for the velocity for the boundary-layer problem. In retrospect, we could have started with the basic assumption that the velocity distribution at the outer edge of the boundary layer approaches asymptotically the inviscid velocity distribution at the wall, and then obtain the corresponding pressure distribution at the outer edge of, and hence across, the boundary layer by means of Bernoulli's equation. However, this second argument is apparently less clear physically.

We cannot expect this fortuitous circumstance for the case of a uniform oncoming flow to occur when the oncoming flow is rotational and where the total pressure changes from streamline to streamline. For, if the second argument is followed, we would then have $u \to U$ as the asymptotic velocity condition at the outer edge of the boundary layer since $u = U$ is precisely the inviscid velocity evaluated at the wall for all finite values of ω. But this will mean that $u \to U$ is a good first approximation to the asymptotic boundary condition for the velocity for all finite ω, which is certainly not true far downstream where the rotational velocity component ωy of the mainstream eventually dominates the uniform flow component U if the boundary layer grows in thickness with distance downstream from the leading edge. The second argument may thus be eliminated as being unsatisfactory. On the other hand, if the boundary layer is thin, the assumptions of Prandtl–Blasius have a definite physical appeal and would constitute a reasonable starting-point for an investigation in the rotational oncoming flow case.

Thus Glauert (1957) begins by assuming the applicability of the Prandtl–Blasius assumptions (as listed above) when the mainstream contains vorticity (see also Li 1955). He then proceeds to deduce that the outward displacement of the external streamlines by the boundary layer causes a reduction in the tangential velocity at the outer edge of the boundary layer when compared to the undisturbed velocity at the same geometrical point in the flow, or, asymptotically,

$$u \to U + \omega(y - \delta^*) \quad \text{as} \quad y \to \infty, \tag{1.1}$$

where δ^* is the displacement thickness of the boundary layer. A different assumption was introduced by Li (1956) to the effect that the tangential velocity at the outer edge of the boundary layer approaches asymptotically the undisturbed velocity near the wall, or

$$u \to U + \omega y \quad \text{as} \quad y \to \infty. \tag{1.2}$$

This assumption then leads automatically to the result that a favourable pressure gradient is induced along the plate that is proportional to the product of ω and the normal displacement velocity at the outer edge of the boundary layer.

Murray (1961) was the first to attempt a resolution of the conflicting claims by examining the respective flows due to displacement thickness (or, mathematically, by matching the boundary layer flow to an outer flow). His work appears to substantiate the assumption of Li (1956). However, further reflexion reveals that his conclusion has been derived prematurely. Like that of Li (1955, 1956) and Glauert (1957), Murray's analysis for the boundary layer (which he carries out in parabolic co-ordinates) is essentially a perturbation analysis about the Blasius solution for small values of the vorticity number $\xi = \omega(\nu x/U^3)^{\frac{1}{2}}$, where ν is the kinematic viscosity. Since ξ contains the factor $\sqrt{x}$ in the numerator, such a perturbation analysis with ξ as the expansion parameter would not be uniformly valid far downstream along the plate ($x \to \infty$). Hence the analysis must be restricted to the region

$$x_L \leqslant x \leqslant x_U, \tag{1.3}$$

where the lower bound x_L must be much greater than ν/U in order for the boundary-layer approximations to apply, and the upper bound x_U must be much smaller than $U^3/\nu\omega^2$ in order for the perturbation analysis to be valid. This means that the flow due to displacement thickness as determined by Murray is based on the information from this limited region only. Such an analysis is incomplete from the physical standpoint because a larger disturbance can be created by the displacement thickness downstream of the region (1.3) than that predicted by Murray, especially if the boundary layer continues to grow in thickness.

An analysis for the boundary layer downstream of the region (1.3), based on assumption (1.2), has been supplied by Ting (1960). Ting gives an asymptotic analysis for large values of the vorticity number† by perturbing about a Couette-type flow $u_c = \omega y$, $v_c = 0$ and $p_c = P$ (constant). He points out that the deviations from this basic flow need not be small, as may be seen by writing his expansion for the pressure to the first two terms as

$$p = P - \beta\rho U(\nu x \omega^2)^{\frac{1}{3}}, \tag{1.4}$$

where β is a constant and ρ is the density. The second term, being the induced pressure due to the presence of the boundary layer, becomes larger than the primary pressure P when $x \to \infty$. This limit is consistent with $\xi \to \infty$ with fixed non-zero U, ν and ω. This means that the superposition of a uniform flow of

† The small vorticity number case may be interpreted qualitatively as occurring when $\delta \ll U/\omega$, where δ is the boundary-layer thickness; since this occurs when the uniform flow dominates the rotational flow component at $y = \delta$, $\delta = O(\sqrt{\nu x/U})$, so that $\omega\delta \ll U$ implies $\xi \ll 1$. The large vorticity number case may be interpreted as occurring when $\delta \gg U/\omega$ or when the rotational flow dominates the uniform flow component at $y = \delta$. In this region, the condition that the inertia and viscous forces are of the same order of magnitude gives $\delta = O(\nu x/\omega)^{\frac{1}{3}}$, so that $\omega\delta \gg U$ implies $\xi^{\frac{2}{3}} \gg 1$.

velocity U, no matter how small but greater than zero, upon the prevailing Couette-type flow would result in infinite suction at sufficiently large distances downstream from the leading edge. Clearly, before this can be accepted, it must be shown that this behaviour is compatible with the flow external to the boundary layer and with the flow below the plate since the induced pressure, if it exists, must be due to the *interaction* between the boundary layer and the external flow. Such a large negative pressure must necessarily affect the rest of the outer flow and it is important to ascertain that the upstream kinematical and dynamical boundary conditions particularly are not violated.

In lieu of these investigations, Ting examined, on an *ad hoc* basis, the effect on the boundary layer of an assumed pressure gradient of the form

$$\frac{dp}{dx} = -\tfrac{1}{3}\beta\rho U\left(\frac{\nu\omega^2}{x^2}\right)^{\frac{1}{3}}.$$

This pressure gradient is thus a primary effect and acts on the boundary layer as an 'applied' pressure gradient in the ordinary sense of boundary-layer theory. The only difference here is that β is unknown and must be determined in the course of the analysis.† When interpreted in this sense, the solution given by Ting is just one member of an infinity of possible solutions.

This *ad hoc* solution does not mitigate its usefulness, however, since any solution at this early stage would be of value in providing some insight into the complex features of the interaction phenomenon. From this standpoint we felt justified in presenting another *ad hoc* solution, based essentially on the Prandtl–Blasius assumptions, that is valid in the region downstream of (1.3). In particular, we shall first present an asymptotic solution for large vorticity numbers; then an approximate solution for arbitrary vorticity numbers will be given that bridges the gap between the solution of Glauert for small vorticity numbers and the present asymptotic solution.

2. Formulation of the problem

A. *The inviscid flow past the plate*

In order to appreciate the inherent difficulties of the viscous case, we shall give an extended description of the inviscid case. The inviscid flow past the plate is given by

$$u_\infty = U+\omega y, \quad v_\infty = 0, \quad p_\infty = P. \tag{2.1 a, b, c}$$

The corresponding stream function Ψ_∞ is given by

$$\Psi_\infty = Uy+\tfrac{1}{2}\omega y^2, \tag{2.2}$$

and the streamline pattern is shown in figure 1.

It is seen that, with the exception of the singular streamline $\Psi_\infty = -U^2/2\omega$ where the velocity vanishes, there are two branches corresponding to any given streamline $\Psi_\infty = \text{const.}$ as given by the relation

$$y = -\frac{U}{\omega} \pm \left(\frac{U^2}{\omega^2}+2\frac{\Psi_\infty}{\omega}\right)^{\frac{1}{2}}. \tag{2.3}$$

† It should be emphasized that, while the gradient of pressure appears in the governing equation for the boundary layer, the pressure *itself* is dynamically related to the velocity through Bernoulli's equation and hence to the asymptotic boundary condition for the velocity at the outer edge of the boundary layer.

In particular, for $\Psi_\infty = 0$, the upper branch is on $y = 0$ and the lower branch on $y = -2U/\omega$. The magnitude of the velocity is constant, but opposite in direction, along the two branches of any given streamline.

As may be seen from figure 1, part of the flow below the plate is in the direction of the positive x-axis and part of it is in the opposite direction. This is one apparent reason for the difficulty in posing a valid viscous flow model for the region below the plate.

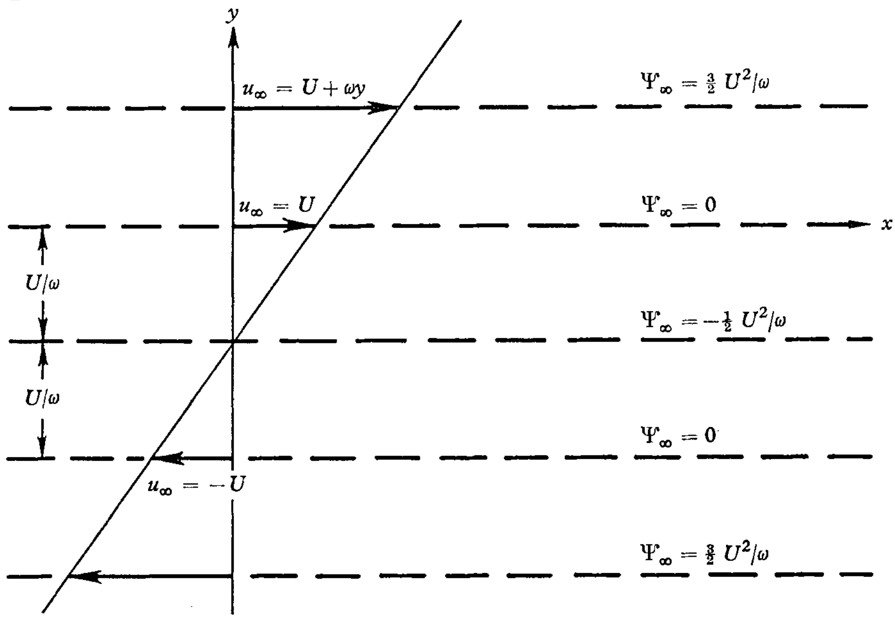

FIGURE 1. Streamline pattern for the undisturbed flow.

B. *Bernoulli's equation for a flow with constant vorticity*

It will be expedient to derive the appropriate Bernoulli equation that is applicable to a region where the vorticity is constant everywhere in the flow (or where viscous diffusion is negligible). The Navier–Stokes equations for the steady flow of an incompressible viscous fluid are

$$\nabla . \mathbf{q} = 0, \tag{2.4a}$$

$$\mathbf{q} \times \mathbf{\Omega} = -\nabla H + \nu \nabla \times \mathbf{\Omega}, \tag{2.4b}$$

where $\mathbf{\Omega}$ is the vorticity vector defined by $\mathbf{\Omega} = -\nabla \times \mathbf{q}$, $\mathbf{q}$ is the velocity vector with magnitude $q = (u^2+v^2)^{\frac{1}{2}}$, and $H = p/\rho + \frac{1}{2}q^2$ is the total head. All other quantities are defined as in § 1. For two-dimensional flow, (2.4*a*) is satisfied by $\mathbf{q} = \nabla\Psi \times \mathbf{k}$, where $\mathbf{k}$ is the unit vector normal to the (x, y)-plane in a right-handed system of axes. Hence, by setting $\mathbf{\Omega} = \omega\mathbf{k}$, (2.4*b*) reduces to

$$\nabla(H - \omega\Psi) = 0,$$

which gives upon integration

$$H - \omega\Psi = \text{const.} \tag{2.5}$$

(2.5) shows that in a flow where the vorticity is everywhere constant, the quantity $(H - \omega\Psi)$ is a universal constant in the flow field. Also, it expresses explicitly that the total head is constant for any given streamline, but varies from streamline to streamline in a rotational flow field. (A relation similar to (2.5) is given in Lamb (1932) for bodies moving in a fluid previously endowed with uniform vorticity.) If $H = H_0 =$ const. on $\Psi = 0$, then the required Bernoulli form is

$$\tfrac{1}{2}(u^2+v^2)+p/\rho = H_0+\omega\Psi. \tag{2.6}$$

C. *The governing equations and boundary conditions for the flow in the boundary layer*

When a semi-infinite flat plate is immersed in a Couette-type flow ($u_c = \omega y$, $v_c = 0$ and $p_c = P$) of a viscous fluid, there is no diffusion of vorticity from the plate, and, hence, there is no boundary layer formed on either side of the plate. This picture is changed when a uniform flow is superposed upon the given flow since vorticity is necessarily diffusing from the plate and convected downstream above the plate and in an unknown manner below the plate. Thus the formulation of a boundary layer is physically plausible for the top of the plate, but not for the bottom. We shall henceforth restrict our considerations to the boundary-layer flow above the plate.

We now assume that the thickness of the boundary layer is very small compared to the distance from the leading edge, so that the outer flow streamlines remain nearly parallel to the plate far downstream. The effect of this thin layer upon the external pressure field is then assumed to be a secondary effect (with the exception of the region near the leading edge). Thus, to the first approximation, the Navier–Stokes equations (2.4) reduce to the Prandtl–Blasius boundary-layer equations for a semi-infinite flat plate.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{2.7a}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}. \tag{2.7b}$$

The asymptotic boundary condition for the velocity at the outer edge of the boundary layer that is consistent with this approximate system of equations may be derived from (2.6) as follows. We assume that the upstream flow conditions are also undisturbed by the presence of the boundary layer,† so that, by virtue of (2.1) and (2.2), $H_0 = (U^2/2)+(P/\rho)$. Now, according to the above assumptions, we set $p = P$ and neglect v^2 in comparison to u^2 in (2.6) to obtain the following asymptotic condition at the outer edge of the boundary layer:‡

$$u^2 \to U^2+2\omega\Psi \quad \text{as} \quad y\to\infty. \tag{2.8a}$$

This condition and the no-slip conditions at the wall

$$u = 0, \quad v = 0 \quad \text{at} \quad y = 0 \tag{2.8b, c}$$

† More precisely, we are assuming here that any flow disturbance due to the presence of the boundary layer decays at least as rapidly as some negative power of the radial distance from the leading edge.

‡ In order to emphasize the asymptotic nature of this condition, the mathematical notation of $y\to\infty$ has been adopted here although it is implied that y is of the order of the boundary-layer thickness.

complete the mathematical formulation of the present boundary-layer problem. This system will be regarded as a valid first approximation for arbitrary vorticity numbers; the truth of this depends, of course, upon the demonstration of the uniform validity of successive approximations. In the following we confine ourselves to the first approximation.

3. Asymptotic solution for large vorticity numbers

Since the stream function appears in the boundary condition (2.8a), it will be more convenient to use the von Mises co-ordinates (X, Ψ) instead of (x, y). They are related by the transformations

$$x = X, \quad dy = d\Psi/u. \tag{3.1a, b}$$

In terms of these new co-ordinates, (2.7) assume the usual form

$$\frac{\partial u}{\partial X} = \frac{\nu}{2}\frac{\partial^2 u^2}{\partial \Psi^2}. \tag{3.2}$$

The associated boundary conditions are

$$u = 0 \quad \text{at} \quad \Psi = 0 \quad \text{and} \quad u^2 \to 2\omega\Psi + U^2 \quad \text{as} \quad \Psi \to \infty. \tag{3.3a, b}$$

We now assume that, for sufficiently large values of the vorticity number ξ, the flow described by this approximate system does not deviate appreciably from the Couette-type flow $[u_c = \omega y$ or $u_c = \sqrt{(2\omega\Psi)}]$. We thus look for an asymptotic solution of the form

$$u = \sqrt{(2\omega\Psi)}\,[1 + f(X, \Psi)], \tag{3.4}$$

where $|f| \ll 1$ for large ξ. The first two terms of this asymptotic solution can be readily obtained by substituting (3.4) into (3.2) and neglecting terms of order f^2; this gives

$$\frac{\partial F}{\partial X} = \nu\sqrt{(2\omega\Psi)}\frac{\partial^2 F}{\partial \Psi^2}, \tag{3.5}$$

where $F = \Psi f$. This equation admits a similar solution of the form $F = F(\eta)$, where $\eta = (2/9\omega\nu^2 x^2)^{\frac{1}{3}}\,\Psi$ and F must satisfy

$$F'' + \sqrt{\eta}\,F' = 0 \tag{3.6a}$$

and the boundary conditions

$$F = 0 \quad \text{at} \quad \eta = 0 \quad \text{and} \quad F \to U^2/4\omega \quad \text{as} \quad \eta \to \infty. \tag{3.6b, c}$$

The solution is easily obtained in the form

$$F = \tfrac{1}{4}(\tfrac{2}{3})^{\frac{2}{3}}\frac{1}{\Gamma(\frac{5}{3})}\frac{U^2}{\omega}\int_0^\eta e^{-\frac{2}{3}\eta^{\frac{3}{2}}}\,d\eta, \tag{3.7}$$

where Γ denotes the gamma function.

At the outer edge of the boundary layer (i.e. where viscous diffusion is exponentially small as may be verified by using (3.7)), $\eta = O(1)$, so that $\Psi = O(\omega\nu^2 x^2)^{\frac{1}{3}}$. Since $F = O(U^2/\omega)$, we see that $f = O(\xi^{-\frac{4}{3}})$ and hence the neglected term f^2 is of order $\xi^{-\frac{8}{3}}$, which justifies the asymptotic nature of the solution.

The local skin friction τ_w is given by

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu \frac{\partial}{\partial \Psi}\left(\frac{u^2}{2}\right)\bigg|_{\Psi=0}, \tag{3.8}$$

so that, by using (3.4) and (3.7), we have (μ is the viscosity)

$$\tau_w \sqrt{\frac{x}{\rho\mu U^3}} = \xi + 0{\cdot}256\xi^{-\frac{1}{3}} + O(\xi^{-\frac{5}{3}}). \tag{3.9}$$

We may define a displacement thickness δ^* as follows. The substitution of (2.8*a*) into (3.1*b*) gives, by a single quadrature,

$$y = B + \frac{1}{\omega}(U^2 + 2\omega\Psi)^{\frac{1}{2}}. \tag{3.10}$$

The constant of integration B may be determined such that $y = \delta^*$ on $\Psi = 0$; this gives $B = \delta^* - U/\omega$, so that (3.10) may be written as

$$\Psi = U(y-\delta^*) + \tfrac{1}{2}\omega(y-\delta^*)^2, \tag{3.11a}$$

or as

$$u = (U^2 + 2\omega\Psi)^{\frac{1}{2}}$$
$$= U + \omega(y-\delta^*). \tag{3.11b}$$

These results are consistent with Glauert's deduction that the streamlines at the outer edge of the boundary layer are displaced outward from the plate by an amount δ^*—just as in the case without external vorticity in the mainstream. By substituting (3.4) into (3.11*b*) and solving for δ^*, we have asymptotically

$$\delta^* = \frac{U}{\omega}[1 + O(\xi^{-\frac{2}{3}})], \tag{3.12}$$

so that the displacement thickness tends to a constant as $\xi \to \infty$.

4. Approximate calculation for arbitrary vorticity numbers

We now show that, by the application of the approximate Karman-integral procedure, a simple theory can be evolved for arbitrary vorticity numbers.

By performing the usual operations to (2.7), with the aid of (2.8), we have

$$u_\delta \frac{d}{dx}\int_0^\delta u\,dy - \frac{d}{dx}\int_0^\delta u^2\,dy = \nu\left(\frac{\partial u}{\partial y}\right)_{y=0} - \nu\omega, \tag{4.1}$$

where we have taken $y = \delta(x)$ as the outer edge of the boundary layer and

$$u_\delta = (U^2 + 2\omega\Psi_\delta)^{\frac{1}{2}} \tag{4.2}$$

as the approximate boundary condition for the tangential velocity at $y = \delta$.

We shall likewise find it convenient to use the von Mises transformation in conjunction with (4.1). We thus have

$$u_\delta \frac{d}{dx}\int_0^{\Psi_\delta} d\Psi - \frac{d}{dx}\int_0^{\Psi_\delta} u\,d\Psi = \nu \frac{\partial}{\partial \Psi}\left(\frac{u^2}{2}\right)\bigg|_{\Psi=0} - \nu\omega, \tag{4.3}$$

where δ is related to Ψ_δ by

$$\delta = \int_0^{\Psi_\delta} d\Psi/u. \tag{4.4}$$

We now approximate u across the boundary layer by the expression

$$u = u_\delta\left(\frac{\Psi}{\Psi_\delta}\right)^{\frac{1}{2}}. \tag{4.5}$$

This assumed form satisfies the conditions $u = 0$ at $\Psi = 0$ and $u = u_\delta$ at $\Psi = \Psi_\delta$. Hence, by using (4.2) and (4.5), (4.3) reduces by a straightforward calculation to

$$\Psi_\delta(U^2+2\omega\Psi_\delta)^{-\frac{1}{2}}\frac{d\Psi_\delta}{dx} = \frac{3\nu}{2}, \tag{4.6}$$

which may be easily integrated to give $\Lambda = \omega\Psi_\delta/U^2$ in terms of the vorticity number ξ as

$$(\Lambda-1)(2\Lambda+1)^{\frac{1}{2}} = \tfrac{9}{2}\xi^2, \tag{4.7}$$

where the constant of integration has been evaluated by taking $\Psi_\delta = 0$ for $x = 0$ (i.e. by assuming that the boundary layer has zero thickness at the leading edge). (4.7) represents our main approximate result—from it we can calculate δ, δ^*, the normal component of velocity v_δ at the outer edge of the boundary layer, and τ_w as a function of ξ.

By expanding (4.7) for small and large ξ, we obtain

$$\Lambda = \begin{cases} \sqrt{3}\,\xi+\xi^2+O(\xi^3) \quad (\xi \ll 1), & (4.8a)\\ \tfrac{1}{2}9^{\frac{2}{3}}\xi^{\frac{4}{3}}+\tfrac{1}{2}+O(\xi^{-\frac{4}{3}}) \quad (\xi \gg 1). & (4.8b)\end{cases}$$

By virtue of (4.4) and (4.5), δ is given in non-dimensional form by

$$D \equiv \delta\left(\frac{U}{\nu x}\right)^{\frac{1}{2}} = \frac{2\Lambda}{\xi(1+2\Lambda)^{\frac{1}{2}}}, \tag{4.9}$$

which may be expanded, with the aid of (4.8), to give

$$D = \begin{cases} 2\sqrt{3}-4\xi+O(\xi^2) \quad (\xi \ll 1),\\ 9^{\frac{1}{3}}\xi^{-\frac{1}{3}}+O(\xi^{-3}) \quad (\xi \gg 1).\end{cases}$$

By virtue of (4.2), (4.9) and (3.11*b*), δ^* is given in non-dimensional form by

$$D^* \equiv \delta^*(U/\nu x)^{\frac{1}{2}} = \xi^{-1}\{1-(1+2\Lambda)^{-\frac{1}{2}}\}, \tag{4.10}$$

which may be expanded, with the aid of (4.8), to give

$$D^* = \begin{cases} \sqrt{3}-\tfrac{7}{2}\xi+O(\xi^2) \quad (\xi \ll 1), & (4.11a)\\ \xi^{-1}-9^{-\frac{1}{3}}\xi^{-\frac{5}{3}}+O(\xi^{-3}) \quad (\xi \gg 1). & (4.11b)\end{cases}$$

(4.11*b*) also may be written as

$$\delta^* = \frac{U}{\omega}\,[1-9^{-\frac{1}{3}}\xi^{-\frac{2}{3}}+O(\xi^{-\frac{4}{3}})],$$

which agrees in form with (3.12).

From (2.7*a*) we have

$$v_\delta = -\int_0^\delta \frac{\partial u}{\partial x}dy = -\frac{d\Psi_\delta}{dx}+u_\delta\frac{d\delta}{dx},$$

so that, by virtue of (4.2), (4.6) and (4.9), we have

$$v_\delta^* \equiv v_\delta\left(\frac{x}{U\nu}\right)^{\frac{1}{2}} = \frac{3}{2}\frac{\xi}{\Lambda(1+2\Lambda)^{\frac{1}{2}}},$$

which may be expanded, with the aid of (4.8), to give

$$v_\delta^* = \begin{cases} \tfrac{1}{2}\sqrt{3}-2\xi+O(\xi^2) \quad (\xi \ll 1),\\ \dfrac{1}{3\xi}-\tfrac{2}{3}9^{-\frac{2}{3}}\xi^{-\frac{7}{3}}+O(\xi^{-\frac{11}{3}}) \quad (\xi \gg 1).\end{cases}$$

Finally, by virtue of (4.2), (4.5) and (3.8), the non-dimensional skin friction is given by

$$\tau_w\left(\frac{x}{\rho\mu U^3}\right)^{\frac{1}{2}} = \xi\left(1+\frac{1}{2\Lambda}\right),$$

which may be expanded, with the aid of (4.8), to give

$$\tau_w\left(\frac{x}{\rho\mu U^3}\right)^{\frac{1}{2}} = \begin{cases} 0{\cdot}289+0{\cdot}833\xi+O(\xi^2) & (\xi \ll 1), \\ \xi+0{\cdot}231\xi^{-\frac{1}{3}}+O(\xi^{-\frac{5}{3}}) & (\xi \gg 1). \end{cases}$$

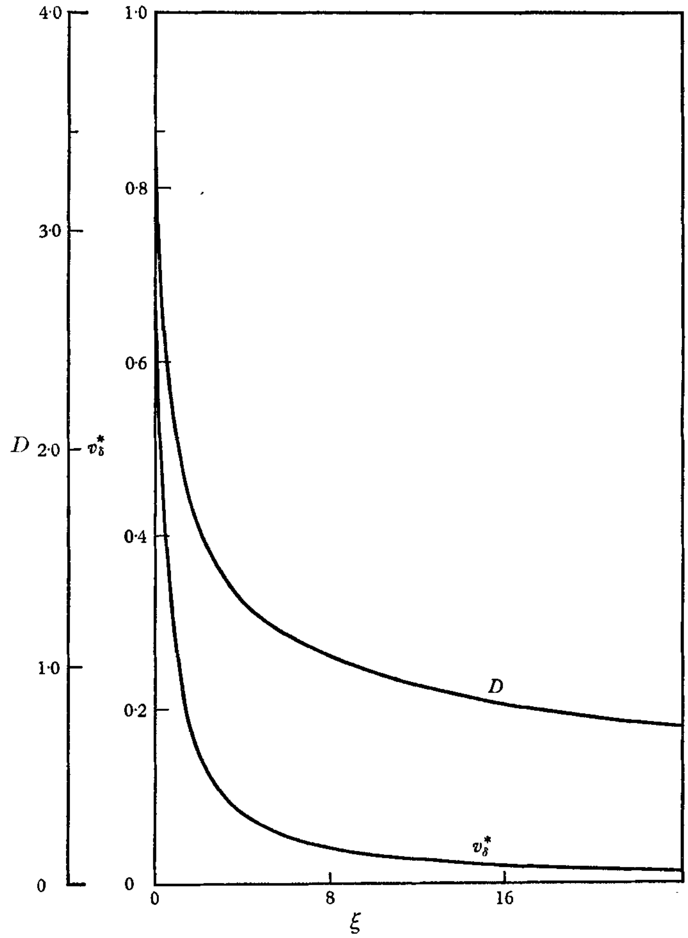

FIGURE 2. Variations of D and v_δ^* with vorticity number ξ.

Glauert's result for the skin friction for $\xi \ll 1$ is

$$\tau_w\left(\frac{x}{\rho\mu U^3}\right)^{\frac{1}{2}} = 0{\cdot}332+0{\cdot}795\xi+O(\xi^2),$$

so that our approximate results for $\xi \ll 1$ and $\xi \gg 1$ (by comparison with (3.9)) are reasonably accurate.

The variations of D and v_δ^* for arbitrary ξ are shown in figure 2. It is seen that both quantities decrease steadily with increasing ξ from their values at $\xi = 0$ so that the initial assumptions that the boundary layer is thin and the outer flow streamlines remain nearly parallel to the plate are justified *a posteriori* for arbitrary ξ.

5. Discussion

The present solutions are based on the crucial assumption that the presence of a thin boundary layer does not affect the external pressure field at sufficiently large distances from the leading edge to the first approximation and for arbitrary vorticity numbers. If it can be rigorously ascertained that the constant-pressure Couette-type flow is the proper limiting solution as $\xi \to \infty$, then the present theory has some measure of truth. However, the result of Ting for the pressure shows that the limiting solution may not be unique since the convergence of p is not uniform as $\xi \to \infty$;† hence the actual state-of-affairs remains in doubt.

It has been suggested that a logical way of resolving this conflict is to determine whether the respective asymptotic solutions for large ξ can be matched to a suitable outer flow that does not exclude the flow below the plate.‡ This is reasonable because we are dealing with a problem of interaction where the conditions of one region and those of the adjacent region are interdependent. However, this would be a formidable task at present; and since the matching procedure is not completely understood for the simpler case of flow without external vorticity, a satisfactory resolution is not likely in the foreseeable future. Nevertheless, we are confident that the present solutions, along with all the previous solutions, will aid immeasurably in completing our understanding of the effect of external vorticity on the flow field when viewed collectively with future work on the problem.

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† Suppose $\xi \to \infty$ occurs when $U \to 0$ for fixed non-zero ω, ν and x. Then, if we fix ρ also, (1.4) gives $p \to P$ as $U \to 0$. But the convergence is not uniform in x—i.e., given a positive number ϵ, we can always find an $x = x_0$ such that $|p-P| > \epsilon$. We have merely to take $\epsilon = P/2$ and $x_0 = (P^3/\beta^3\rho^3U^3\nu\omega^2)$.

‡ It is emphasized here that this matching be accomplished asymptotically at large ξ (or x) since the maximum pressure disturbance due to displacement thickness generally occurs for large x $(U > 0)$.